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ANALYZING THE LOCAL LINDELÖF PROPER FUNCTION AND THE LOCAL PROPER FUNCTION OF DEEP LEARNING IN BITOPOLOGICAL SPACES

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Abstract. It is essential to create new mathematical strategies to deal with everyday problems since they require a lot of data and ambiguity. The best tool for doing this is proper functions, which are the most common mathematical technique. In order to generate suitable functions, we investigate several set operators. A connection between symmetry and certain types of proper functions and their classical topologies can be made. As a result of this symmetry, we can examine the traits and behaviors of traditional topological notions through settings, and vice versa. We describe a new class of proper functions in this paper and launch a preliminary investigation into them. These functions are referred to as pairwise local proper functions and pairwise local Lindelöf proper functions in bitopological spaces. In general topology, we also establish the connection between this new class of proper functions and other classes of generalized functions already in existence. Regarding the new ideas, a number of relationships, necessary and sufficient conditions, examples and counter-examples are provided. In addition, a different argument for the pairwise regularity of a pairwise Hausdorff and pairwise locally compact bitopological space is presented. As part of this research, we also look at the images and inverse images of specific bitopological features under these functions. A few product theorems pertaining to these concepts were finally discovered.

1. Introduction

For us to comprehend and interpret the real world, it is too complex. As a result, attempts are made to create simplified representations of reality. However, these mathematical models are also exceedingly intricate, making it quite challenging to examine them. Therefore, while solving problems, applying standard covers theory based on examples is not always applicable. Numerous mathematical theories have been created to address these issues, including proper functions, indeterminate set theory, and mathematical theory. These theories are weapons against circumstances. All of these hypotheses, it has been discovered, are flawed in different ways. There have been some proposed generalized topological structures. due to the topological space's significance in analysis and in a number of applications. One of the topological space's most significant generalizations is represented by the appropriate functions. The generation of new forms of covers and the important topological characteristics of the new covers depend critically on the open covers, as we know from general topology. In the field of metric spaces, Vainstein[29] pioneered the concept of the class of proper functions in 1947. Proper functions were independently introduced and researched in the context of locally compact spaces. Later, a number of

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mathematicians focused on locally compact and demonstrated certain findings, including: A compact space is locally compact, while the contrary is not always true; a locally compact space has a closed subset for each, additionally locally compact space need not be continuous to be considered locally compact. Two arbitrary topologies on a non-empty set were systematically investigated by Kelly in 1963, which led to the beginning of a new theory, known as the theory of bitopological spaces [15]. For the analysis of non-symmetric functions that introduce two arbitrary topologies on a non-empty set, this novel concept of bitopological spaces has shown to be quite useful. The generalization and extension of key classical topology concepts and findings to bitopological spaces has also been done in this study. The definitions of selected separation qualities from conventional topology have been extended in bitopological spaces and given new names, including compactness, local compactness, lindelöf, local lindelöf, separation axiomes, kinds of functions, and other topics. Afterwards in 1967, N. Krolevec [17] created a locally perfect mapping and provided certain attributes. When creating pairwise locally compact in bitopological spaces in 1972, Reilly [22] gave several features. Subsequently, in 1979, D. Somasundaram and G. Balasubramanian presented locally lindelöf spaces and offered certain characteristics [25]. In 2020, H. Singh and S. Mishra [24] provide a new definition of pairwise locally compact in bitopological space. In the latter part of 2021, N. Abualkishik and H. Hdeib [1] accumulate pairwise locally compact and pairwise locally lindelöf space while offering specific advantages. Using the concept of the proper functions and studying its features, this investigation generalized new forms of proper functions. Additionally, how it relates to previous ideas that have been established. Furthermore, a new category of functions, such as the local lindelöf proper function and the local proper function, are defined. Figuring out how they relate to one another, offering numerous examples and qualities that are relevant to this function, and this function will serve as a beginning point for research into the function's many potential futures.

2. Preliminary Statements and Essential Definitions

In the sections that follow, we give the basic definitions and theorems that we will employ to bolster our main conclusions. To set the stage for our investigation, we will refer to bitopological spaces as "spaces" throughout this work.

We'll start by going over the major concepts and conclusions that will be applied to the entire project.

Definition 2.1. [7] The definition of pairwise continuous refers to a function $\Theta : (G, \alpha_1, \alpha_2) \rightarrow (K, \beta_1, \beta_2)$, if both $\Theta_1 : (G, \alpha_1) \rightarrow (K, \beta_1)$ and $\Theta_2 : (G, \alpha_2) \rightarrow (K, \beta_2)$ are continuous functions.

Definition 2.2. [7] If the two the functions $\Theta_1 : (G, \alpha_1) \rightarrow (K, \beta_1)$ and $\Theta_2 : (G, \alpha_2) \rightarrow (K, \beta_2)$ are closed functions, the function $\Theta : (G, \alpha_1, \alpha_2) \rightarrow (K, \beta_1, \beta_2)$ is called out to as pairwise closed. According to this L_1 is closed in α_1,then Θ (L_1) is closed in β_1, and if L_2 is closed in τ_2, then $\Theta(L_2)$ is closed in β_2.

Definition 2.3. [15] A cover $\hat{E}$ of the bitopological space (G, α_1, α_2) is called α_1, α_2-open if $\hat{E} \subset \alpha_1 \cup \alpha_2$.

Additionally, $\hat{E}$ is referred to as pairwise open if it has at least one nonempty member of α_2.

Definition 2.4. [11] Any pairwise open cover of a bitopological space that has a finite subcover is known to as pairwise compact.

Definition 2.5. [22] If (G, α_1, α_2) is a bitopological space then α_1 is locally compact with regard to α_2 if every point of (G, α_1, α_2) has a α_1- open neighbourhood who's α_2- closure is pairwise compact.

Definition 2.6. [22] If both $\alpha_1$and α_2 are locally compact with respect to each other, then (G, α_1, α_2) is pairwise locally compact. In relation to α_1, α_2 is locally compact.

Theorem 2.7. *[11] The statements that follow are analogous if (G, α_1, α_2) is pairwise Hausdorff:*

(a) In relation to α_1, α_2 is locally compact.

(b) For every point $g \in G$ and each $\alpha_1$open set E containing g exists a α_1 open set$T$ such as that $g \in T \subset \alpha_2 clT \subset E$ and α_2 cl T is pairwise compact.

Corollary 2.8. *[11] The pairwise regularity of (G, α_1, α_2) depends on whether it is pairwise Hausdorff and pairwise locally compact.*

Theorem 2.9. *[11] Pairwise locally compact spaces are always pairwise compact spaces, although the reverse is not necessarily true.*

Theorem 2.10. *[24] When (K, α_1, α_2) is a subset of (G, α_1, α_2) and (G, α_1, α_2) is a pairwise locally compact space,(K, α_1, α_2) additionally becomes pairwise locally compact.*

Definition 2.11. [11] Any pairwise open cover of a bitopological space that possesses a countable subcover is referred to as a pairwise lindelöf.

Definition 2.12. [1] If (G, α_1, α_2) is a bitopological space then α_1 is locally lindelöf with regard to α_2 if every point of (G, α_1, α_2) has a α_1- open neighbourhood who's α_2- closure is pairwise lindelöf.

Definition 2.13. [17] A mapping $\Theta : (G, \alpha) \to (K, \beta)$ is considered to be locally perfect if, for any point g, there is a neighborhood E whose image is closed in K and whose $\Theta \backslash [E]$ is perfect.

Definition 2.14. [2] Whenever $\Theta : (G, \alpha) \to (K, \beta)$ is continuous, closed, and for all $g \in (K, \beta)$, $\Theta^{-1}(g)$ is lindelöf, then the function $\Theta$is commonly referred to as Lindelöf perfect function.

Theorem 2.15. *[11] If (G, α_1, α_2) is a pairwise Hausdroff space, then every α_b-compact subset is α_n-closed $(b \neq n, b, n = 1, 2)$.*

Theorem 2.16. *[11] A pairwise compact space's α_b-closed proper subset $(b \neq n, b, n = 1, 2)$ is α_n-compact.*

Definition 2.17. [2] A function $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ is called pairwise strongly function, if for each pairwise open cover $\underset{\sim}{E} = \{E_\mu : \mu \in \Delta\}$, it exists $\underset{\sim}{T} = \{T_\delta : \delta \in \Gamma\}$ of (K, β_1, β_2), that way $\Theta^{-1}(T) \subseteq \bigcup \{E_\mu : \mu \in \Delta_1, \Delta_1 \subset \Delta, \text{ finite}\}, \forall t \in T$.

Definition 2.18. [2] A bitopological space (G, α_1, α_2) is called pairwise weakly compact, as if for each finite pairwise open cover $\underset{\sim}{N}$ of (G, α_1, α_2), There is a pairwise open finite subfamily$\underset{\sim}{M}$ of $\underset{\sim}{N}$, such that $(G, \alpha_1, \alpha_2) = \overline{\bigcup \left\{ M/M \in \underset{\sim}{M} \right\}}^{\alpha_O}$, $O = 1, 2$.

Definition 2.19. [2]A bitopological space (G, α_1, α_2) is called pairwise weakly lindelöf, as if for each countable pairwise open cover $\underset{\sim}{N}$ of (G, α_1, α_2), there is a pairwise open finite subfamily $\underset{\sim}{M}$ of $\underset{\sim}{N}$, such that $(G, \alpha_1, \alpha_2) = \overline{\bigcup \left\{ M/M \in \underset{\sim}{M} \right\}}^{\alpha_O}$, $O = 1, 2$.

3. Different Classes For Pairwise Proper Functions

Here, we present a brand-new definition for proper functions in bitopological spaces and demonstrate how they relate to other functions.

Definition 3.1. A function $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ is called pairwise proper function, if Θ is pairwise continuous, pairwise closed, and for each $k \in K$, $\Theta^{-1}(k)$ is pairwise compact.

Definition 3.2. A function $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ is called pairwise lindelöf proper function, if Θ is pairwise continuous, pairwise closed, and for each $k \in K$, $\Theta^{-1}(k)$ is pairwise lindelöf .

Example 3.3. Suppose that $\Theta : (R, \alpha_{ind}, \alpha_{ind}) \longrightarrow (R, \alpha_{ind}, \alpha_{ind})$ is the identity function, where α_{ind} is indiscrete topology, then Θ is pairwise lindelöf proper function. Given that Θ is pairwise continuous, pairwise closed and for every $k \in K$, any open cover $\tilde{E}$ of $\Theta^{-1}(k)$, G is the only non-empty open set in $(R, \alpha_{ind}, \alpha_{ind})$, therefore it definitely includes G. The result is that $\{g\}$ is a countable subcover of $\tilde{E}$ and that $\Theta^{-1}(k)$ is pairwise lindelöf.

Remark 3.4. It is not necessary for the opposite to be true if a function $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ is a pairwise lindelöf proper function.

Proof. The fact that Θ is pairwise continuous, pairwise closed and for each $k \in K$, $\Theta^{-1}(k)$ is pairwise compact, followed by $\Theta^{-1}(k)$ is pairwise lindelöf. Θ is pairwise lindelöf proper function as a result. □

Corollary 3.5. *Assuming $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ is a pairwise lindelöf proper function, then Θ does not necessarily need to be a pairwise proper function.*

Example 3.6. Suppose $\Theta : (R, \alpha_s, \alpha_s) \to (R, \alpha_s, \alpha_s)$ is pairwise lindelöf proper function. Because in $R_{Sorgenfrey}$, Assuming $\tilde{E} = \{[-g; g) : g > 0\}$ be a cover with several countable subcovers. For illustration, T=$\{(-n^*, n^*) : n^* \in N\}$ is a subcover of $\tilde{E}$, each one $k \in R_{Sorgenfrey}$, $\Theta^{-1}(k)$ is countable. The result $\Theta^{-1}(k)$ is pairwise lindelöf. Nevertheless, keep in mind that T=$\{(-n^*, n^*) : n^* \in N\}$is infinite subcollection of $\tilde{E}$ is covered R. Consequently, $\Theta^{-1}(k)$ is not pairwise compact. Therefore, Θ is not pairwise proper function.

Definition 3.7. A function $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ is called pairwise locally proper function, whenever Θ is pairwise continuous, pairwise closed, for each $k \in K$, $\Theta^{-1}(k)$ is pairwise locally compact.

Theorem 3.8. *Every pairwise proper function also occurs to be a pairwise locally proper function, while the reverse is not correct.*

Proof. It is clear that Θ is pairwise continuous, pairwise closed and for each $k \in K$, $\Theta^{-1}(k)$ is pairwise compact, then $\Theta^{-1}(k)$ is pairwise locally compact. It implies that Θ is pairwise locally proper function. □

The example that follows demonstrates that the opposite need not be accurate.

Example 3.9. Suppose $\Theta : (R, \alpha_{dis}, \alpha_{coc}) \rightarrow (R, \alpha_{dis}, \alpha_{coc})$ is identity function. There is no doubt that while Θ is pairwise locally proper function, it is not pairwise proper function.

Definition 3.10. A function $\Theta : (G, \alpha_1, \alpha_2) \rightarrow (K, \beta_1, \beta_2)$ is called pairwise locally lindelöf proper function, whenever Θ is pairwise continuous, pairwise closed, for each $k \in K$, $\Theta^{-1}(k)$ is pairwise locally lindelöf.

Example 3.11. Take $\Theta : (N, \alpha_{dis}, \alpha_{ind}) \rightarrow (N, \alpha_{dis}, \alpha_{ind})$ is identity function. When such happens, Θ is pairwise locally lindelöf proper function.

Example 3.12. Identity function is defined as $\Theta : (R, \alpha_{dis}, \alpha_{ind}) \rightarrow (R, \alpha_{dis}, \alpha_{ind})$.When such happens, Θ is not a pairwise locally lindelöf proper function.

From Theorem 3.8, the proofs of the following theorems flow naturally.

Theorem 3.13. *Each pairwise lindelöf proper function is pairwise locally lindelöf proper function.*

Theorem 3.14. *Every pairwise locally proper function is pairwise locally lindelöf proper function.*

4. New Pairwise Locally Proper FunctionTheorems

In this part, we provide fundamental theorems for pairwise locally proper function and pairwise locally lindelöf proper function in topological spaces and demonstrate how they connect to other spaces.

Theorem 4.1. *For each pairwise locally compact subset of* $(Z, \beta_1, \beta_2) \subseteq (K, \beta_1, \beta_2)$*, while* $\Theta : (G, \alpha_1, \alpha_2) \rightarrow (K, \beta_1, \beta_2)$ *is a pairwise locally proper function, therefore* $\Theta^{-1}(Z, \beta_1, \beta_2)$ *is a pairwise locally compact.*

Proof. Suppose that $\hat{E} = \{E_\mu : \mu \in \Delta\}$ is a pairwise open cover of (G, α_1, α_2). Due to the fact that Θ is a pairwise locally proper function, therefore $\forall k \in K$, $\Theta^{-1}(k)$ is pairwise locally compact, given are a finite subsets Δ_k, Δ_k^* of Δ. Assuming that $\Theta^{-1}(k) \subseteq \bigcup_{\mu \in \Delta_k} \{T_\mu : \mu \in \Delta_k\}$, while $\{T_\mu : \mu \in \Delta_k\}$ is α_1-open neighbourhood wherein $\overline{\{J_\mu : \mu \in \Delta_k^*\}}$ is α_2-compact. Suppose $H_k = K - \Theta\ (G - \bigcup_{\mu \in \Delta_k} T_\mu)$ is a β_1-open neighbourhood containing k, whence $H_k^* = K - \Theta\ (G - \overline{\bigcup_{\mu\in\Delta_k^*} J_\mu})$ is a β_2-compact set containing k. Currently, $\Theta^{-1}(H_k) \subseteq \bigcup_{\mu\in\Delta_k} T_\mu$ is α_1-open neighbourhood while $\Theta^{-1}\ (H_k^*) \subseteq \overline{\bigcup_{\alpha\in\Lambda_y^*} J_\mu}$ is α_2-compact. Because $(Z,\beta_1,\beta_2) \subseteq (K,\beta_1,\beta_2)$ is pairwise locally compact $Z \subset \bigcup_{o=1}^{n}(H_{k_o})$ is a β_1-open neighbourhood containing k, whose $\bigcup_{p=1}^{m}(H_{k_p}^*)$ is a β_2-compact set containing$k$. Therefore, $\Theta^{-1}(Z) \subseteq \bigcup_{o=1}^{n}\Theta^{-1}(H_{k_o})$ is α_1-open neighbourhood whose $\bigcup_{p=1}^{m}\Theta^{-1}(H_{k_p}^*)$ is α_1-compact. Meant to be $\Theta^{-1}(Z)$ is pairwise locally compact. □

We received the following remarks using the same method of proof.

Remark 4.2. Under pairwise locally proper functions, a pairwise locally compact space is inversely invariant.

Remark 4.3. A pairwise locally proper function is a function that is composed of two other pairwise locally proper functions.

Remark 4.4. Over pairwise locally lindelöf proper functions, a pairwise locally lindelöf space is inversely invariant.

Remark 4.5. A pairwise locally lindelöf proper function is created by the composition of two such functions.

Proposition 4.6. *Assuming the pairwise continuous funcions are composed $F\ \circ\Theta$ as follows: $\Theta : (G,\alpha_1,\alpha_2) \overset{onto}{\to} (K,\beta_1,\beta_2)$, $F : (K,\beta_1,\beta_2) \overset{onto}{\to} (Z,\tau_1,\tau_2)$ are a pairwise closed , subsequently, the function $F : (K,\beta_1,\beta_2) \overset{onto}{\to} (Z,\tau_1,\tau_2)$ is pairwise closed.*

Proof. While Q be a β_1-closed in (K,β_1,β_2),therefore $\Theta^{-1}(Q)$ is α_1-closed in (G,α_1,α_2). Given that $F\ \circ\Theta$ is pairwise closed, $F(\Theta(\Theta^{-1}(Q))$ is τ_1-closed in (Z,τ_1,τ_2). It indicates that $F(Q)$ is τ_1-closed in (Z,τ_1,τ_2). Comparable to this, we may demonstrate that if W be a β_2-closed in (K,β_1,β_2), then $F(Q)$ is τ_2-closed in (Z,τ_1,τ_2). As a result, F is a pairwise closed function. □

Theorem 4.7. *Assuming the pairwise continuous funcions are composed $F\ \circ\Theta$ as follows: $\Theta : (G,\alpha_1,\alpha_2) \overset{onto}{\to} (K,\beta_1,\beta_2)$, $F : (K,\beta_1,\beta_2) \overset{onto}{\to} (Z,\tau_1,\tau_2)$ are pairwise locally proper function, then the function $F : (K,\beta_1,\beta_2) \overset{onto}{\to} (Z,\tau_1,\tau_2)$ is pairwise locally proper function.*

Proof. For each $z \in Z$, $F^{-1}(z) = \Theta\ ((F \circ\Theta\ )^{-1}(z))$ is pairwise locally compact, as a result of being $F\ \circ\Theta$ is pairwise locally proper function. Due to the fact that by proposition [4.6], F is pairwise closed. The pairwise locally proper function F is what we discover. □

The following theorem is obtained using the same proof strategy.

Theorem 4.8. *Assuming the pairwise continuous funcions are composed $F\ \circ\Theta$ as follows: $\Theta : (G,\alpha_1,\alpha_2) \overset{onto}{\to} (K,\beta_1,\beta_2)$, $F : (K,\beta_1,\beta_2) \overset{onto}{\to} (Z,\tau_1,\tau_2)$ are pairwise locally lindelöf* *proper function, then the function* $F : (K, \beta_1, \beta_2) \overset{onto}{\to} (Z, \tau_1, \tau_2)$ *is pairwise locally lindelöf proper function.*

Theorem 4.9. *Assuming* $\Theta : (G, \alpha_1, \alpha_2) \overset{onto}{\to} (K, \beta_1, \beta_2)$ *is pairwise closed function,*

Afterwards, For every $(W, \beta_1, \beta_2) \subset (K, \beta_1, \beta_2)$, the restriction $\Theta_W : \Theta^{-1}(W) \to W$ is pairwise closed.

Proof. Suppose $(W, \beta_1, \beta_2) \subset (K, \beta_1, \beta_2)$.Take into account the function $\Theta_1 : (G, \alpha_1) \to (K, \beta_1)$. Make Q be a α_1-closed. Following that $\Theta_W\ (Q \bigcap \Theta^{-1}(W)) = \Theta(Q) \bigcap W$ is β_1-closed in W. Comparably, we may demonstrate that if H a α_2-closed, $\Theta_W\ (H \bigcap \Theta^{-1}(W) =$

$\Theta(H) \bigcap W$ is β_2-closed in W. Therefore $\Theta_W : \Theta^{-1}(W) \to W$ is pairwise closed. □

Theorem 4.10. *Suppose* $\Theta : (G, \alpha_1, \alpha_2) \overset{onto}{\to} (K, \beta_1, \beta_2)$ *is pairwise locally proper function ,*

then for each $(W, \beta_1, \beta_2) \subset (K, \beta_1, \beta_2)$, the restriction $\Theta_W : \Theta^{-1}(W) \to W$ is pairwise locally proper function.

Proof. Theorem 4.9 leads to proof . □

We arrive to the following theorem using the same proof strategy.

Theorem 4.11. *Assume* $\Theta : (G, \alpha_1, \alpha_2) \overset{onto}{\to} (K, \beta_1, \beta_2)$ *is pairwise locally lindelöf proper function, then for any* $(W, \beta_1, \beta_2) \subset (K, \beta_1, \beta_2)$,*the restriction* $\Theta_W : \Theta^{-1}(W) \to W$ *is pairwise locally lindelöf proper function.*

Theorem 4.12. *Assuming that* (G, α_1, α_2)*is a pairwise Hausdroff space, then each* α_o-*locally compact subset is* α_p-*closed* $(o \neq p, o, p = 1, 2)$.

Proof. The theorem 2.15 dictates the proof . □

Theorem 4.13. *A* α_o*-closed proper subset of pairwise locally compact space is* α_p *−locally compact* $(o \neq p, o, p = 1, 2)$.

Proof. The theorem 2.16 dictates the proof. □

Theorem 4.14. *If* $\Theta : (G, \alpha_1, \alpha_2) \overset{onto}{\to} (K, \beta_1, \beta_2)$ *is pairwise locally proper function,*

where (G, α_1, α_2) is pairwise locally compact, and (K, β_1, β_2) is pairwise Hausdorff ,

then Θ is pairwise closed .

Proof. Given that (G, α_1, α_2) is pairwise locally compact,when R is α_1-closed subset of (G, α_1, α_2), it is α_2-locally compact. Considering that Θ is pairwise continuous. $\Theta(R)$ is a β_2-locally compact subset of (K, β_1, β_2). The fact that (K, β_1, β_2) is pairwise Hausdorff means that $\Theta(R)$ is a β_1-closed. The same is true whether T is a α_2-closed subset of (G, α_1, α_2), then $\Theta(T)$ is a β_2-closed subset of (K, β_1, β_2). □

Remark 4.15. In the event $\Theta : (G, \alpha_1, \alpha_2) \overset{onto}{\to} (K, \beta_1, \beta_2)$ is pairwise locally lindelöf proper function, where (G, α_1, α_2) is pairwise locally lindelöf, and (K, β_1, β_2) is pairwise Hausdorff, subsequently Θ is pairwise closed .

Theorem 4.16. *Suppose* $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ *is pairwise continuous function, from a pairwise Hausdorff space* (G, α_1, α_2) *to a pairwise locally compact space* (K, β_1, β_2)*. Consequently, the following statements are equivalent:*

(a) Θ is a pairwise locally proper function,

(b) For each pairwise locally compact subset $(Z, \alpha_1, \alpha_2) \subset (G, \alpha_1, \alpha_2)$

the set $\Theta^{-1}(Z, \alpha_1, \alpha_2)$ is a pairwise locally compact subset of (G, α_1, α_2).

Proof. (a)$\Rightarrow$ (b) :according to theorem 4.1.

(b)$\Rightarrow$(a) : Only demonstrating that Θ is a pairwise closed function is necessary. It is unpleasant $\Theta_1 : (G, \alpha_1) \to (K, \beta_1)$

and $\Theta_2 : (G, \alpha_2) \to (K, \beta_2)$ are closed functions. Suppose Q is a α_1-closed subset of (G, α_1, α_2), and k be a cluster point $\Theta_1(Q)$.Assume that $k \notin \Theta_1(Q)$.Due to (K, β_1, β_2) is pairwise locally compact, there is a β_1-open set O containing k like that $\overline{O}^{\beta_2}$is pairwise compact. Currently, $\Theta_1^{-1}(\overline{O}^{\beta_2} \bigcap \Theta_1(Q)) = \Theta_1^{-1}(\overline{O}^{\beta_2}) \bigcap Q$.

Utilizing (b) $\Theta_1^{-1}(\overline{O}^{\beta_2})$ is pairwise locally compact and Q is a α_2-closed, pairwise locally compact subset. Right now, $\Theta_1 \ (\Theta_1^{-1}(\overline{O}^{\beta_2}) \bigcap Q\ ) = \overline{O}^{\beta_2} \bigcap \Theta_1(Q)$ is a pairwise locally compact subset which is β_1-closed. Currently,$O - \overline{O}^{\beta_2} \bigcap \Theta_1(Q) = H$ is a β_1-open neighbourhood set containing d and $H \bigcap \Theta_1(Q) = \phi$, d is a cluster point, which goes against the statement. Therefore, $d \in \Theta_1(Q)$,It is nasty $\Theta_1(Q)$ is a β_1-closed. This means $\Theta_1 : (G, \alpha_1) \to (K, \beta_1)$ is a closed function. We can demonstrate that using a similar technique. $\Theta_2 : (G, \alpha_2) \to (K, \beta_2)$ are closed function. Consequently, $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ is pairwise closed function . □

We obtain the following theorem using the same proof strategy.

Theorem 4.17. *Suppose* $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ *is pairwise continuous function, from a pairwise Hausdorff space* (G, α_1, α_2) *to a pairwise locally lindelöf space* (K, β_1, β_2)*. Consequently, the following statements are equivalent:*

(a) Θ is a pairwise locally lindelöf proper function,

(b) For each pairwise locally lindelöf subset $(Z, \alpha_1, \alpha_2) \subset (G, \alpha_1, \alpha_2)$

the set $\Theta^{-1}(Z, \alpha_1, \alpha_2)$ is a pairwise locally lindelöf subset of (G, α_1, α_2).

Theorem 4.18. *Take* $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ *is a pairwise continuous bijection function.*

When (K, β_1, β_2) is pairwise Hausdroff space, and (G, α_1, α_2) is pairwise locally compact, afterward Θ is pairwise homeomorphism function.

Proof. It suffices to demonstrate this Θ is pairwise closed. Suppose C is a α_o-closed proper subset of (G, α_1, α_2), as a result C is proper α_p -locally compact, for $o \neq p, o, p = 1, 2$. Because each α_o-closed proper subset of pairwise locally compact space is α_p $-$locally compact for $o \neq p, o, p = 1, 2$, that's why $\Theta\,(C)$ is a α_p - locally compact. Nevertheless (K, β_1, β_2) is pairwise Hausdroff space. Now that $\Theta : (G, \alpha_1, \alpha_2) \overset{onto}{\rightarrow} (K, \beta_1, \beta_2)$ is pairwise locally proper function, where (G, α_1, α_2) is pairwise locally compact, and (K, β_1, β_2) is pairwise Hausdorff, subsequent Θ is pairwise closed, so $\Theta(F)$ is β_O-closed, It is nasty Θ is pairwise homeomorphism function. □

Remark 4.19. Suppose $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ is a pairwise continuous bijection function.When (K, β_1, β_2) is pairwise Hausdroff space, and (G, α_1, α_2) is pairwise locally lindelöf, afterward Θ is pairwise homeomorphism function.

Theorem 4.20. *Take* $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ *is a pairwise strongly onto function, subsequently* (G, α_1, α_2) *is pairwise locally compact, whether* (K, β_1, β_2) *holds.*

Proof. Suppose $\underset{\sim}{N} = \{N_\alpha : \alpha \in \Gamma\}$ is a pairwise open cover (G, α_1, α_2). Due to Θ is pairwise strongly function, a thing exists pairwise open cover $\underset{\sim}{M} = \{M_\gamma : \gamma \in \Psi_k\}$of (K, β_1, β_2), that way $\Theta^{-1}(M) \subseteq \bigcup\{N_\alpha : \alpha \in \Gamma_1,\ \Gamma_1 \subset \Gamma, \text{finite}\}$,$\forall m \in \underset{\sim}{M}$, however, (K, β_1, β_2) is pairwise locally compact, hence, there is $\Psi_1 \subseteq \Psi$, where Ψ_1 is finite, that way

$(K, \beta_1, \beta_2) = \bigcup_{\gamma \in \Psi_{k_1}} M_\gamma$,where $\{M_\gamma : \gamma \in \Psi_k\}$ is β_1-open neighbourhood whose $\overline{\{W_\gamma : \gamma \in \Psi_k^*\}}$ is β_2-compact. Hence $(G, \alpha_1, \alpha_2) = \bigcup_{\gamma \in \Psi_{k_1}} \Theta^{-1}(M_\gamma)$, in which $\{\Theta^{-1}(M_\gamma) : \gamma \in \Psi_k\}$ is α_1-open neighbourhood whose $\overline{\{\Theta^{-1}(W_\gamma) : \gamma \in \Psi_k^*\}}$ is α_2-compact. Thus (G, α_1, α_2) is pairwise locally compact. □

Corollary 4.21. *Take* $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ *is a pairwise strongly onto function, subsequently* (G, α_1, α_2) *is pairwise locally lindelöf, whether* (K, β_1, β_2) *holds.*

Theorem 4.22. *Let* $\Theta : (G, \alpha_1, \alpha_2) \to (K, \beta_1, \beta_2)$ *be a pairwise locally proper function,* $\forall k \in (K, \beta_1, \beta_2)$, $\Theta^{-1}(k)$ *is pairwise countably compact, and* (K, β_1, β_2) *is a pairwise countably compact, then* (G, α_1, α_2) *is so.*

Proof. Suppose $\underset{\sim}{N} = \{N_\epsilon : \epsilon \in \Gamma\}$ is a pairwise open cover (G, α_1, α_2). Due to Θ is a pairwise locally proper function, then $\forall k \in (K, \beta_1, \beta_2)$, $\Theta^{-1}(k)$ is pairwise locally compact. Instances of a finite subsets Γ_k, Γ_k^* of Γ, like that $\Theta^{-1}(k) \subseteq \bigcup_{\gamma \in \Psi_k} \{M_\gamma : \gamma \in \Psi_k\}$, where $\{M_\gamma : \gamma \in \Psi_k\}$ is α_1-open neighbourhood whose $\overline{\{W_\epsilon : \epsilon \in \Gamma_k^*\}}$ is α_2-compact. Suppose $H_k(\epsilon, k) = (K, \beta_1, \beta_2) - \Theta((G, \alpha_1, \alpha_2) - \bigcup_{\gamma \in \Psi_k} M_\gamma)$ is a β_1-open set containing k, and $H_k^*(\epsilon, k) = (K, \beta_1, \beta_2) - \Theta((G, \alpha_1, \alpha_2) - \bigcup_{\gamma \in \Psi_k^*} \overline{W_\gamma : \gamma \in \Psi_k^*})$ is a β_2-compact set containing k, where $\Theta^{-1}(H_k(\epsilon, k)) \subseteq \bigcup_{\gamma \in \Psi_k} M_\gamma$ is α_1-open neighbourhood whose

$\Theta^{-1}\left(H_k^*(\epsilon,k) \subseteq \bigcup_{\gamma\in\Psi_k^*} \overline{W_\gamma : \gamma \in \Psi_k^*}\right.$ is α_2-compact. Let $\{H_k(\epsilon,k)\ :\ k\in\ K\}\bigcup\{H_k^*(\epsilon,k)\ :\ k\in\ K\}$ be a pairwise countable compact cover of (K,β_1,β_2). Because (K,β_1,β_2) is pairwise countably compact, it has pairwise finite subcover say: $\{H_{k_O}\}_{O=1}^{n} and\ \{H_{k\ P}^*\}_{P=1}^{m}$,

so $(G,\alpha_1,\alpha_2) = \bigcup_{o=1}^{n} f^{-1}(O_{y_i}) \bigcup \bigcup_{p=1}^{m} f^{-1}(\overset{*}{O}_{y_j})$. Therefore, (G,α_1,α_2) is a pairwise countably compact . □

Corollary 4.23. *Let* $\Theta : (G,\alpha_1,\alpha_2) \to (K,\beta_1,\beta_2)$ *be a pairwise locally lindelöf proper function,* $\forall k \in (K,\beta_1,\beta_2)$, $\Theta^{-1}(k)$ *is pairwise countably compact, and* (K,β_1,β_2) *is a pairwise countably compact, then* (G,α_1,α_2) *is so.*

According to the next theorem, pairwise paracompactness is an inverse invariant under pairwise locally proper function.

Theorem 4.24. *Let* $\Theta : (G,\alpha_1,\alpha_2) \to (K,\beta_1,\beta_2)$ *be a pairwise locally proper function,*

and (K,β_1,β_2) is a pairwise paracompact, then (G,α_1,α_2) is so.

Proof. Let $\underset{\sim}{N} = \{N_\epsilon : \epsilon\in\Gamma\}$ be a pairwise open cover of (G,α_1,α_2), since Θ is a pairwise locally proper function, then $\forall k \in (K,\beta_1,\beta_2)$, $\Theta^{-1}(k)$ is pairwise locally compact, there exists a finite subsets Γ_k, Γ_k^* of Γ, such that $\Theta^{-1}(k) \subseteq \bigcup_{\gamma\in\Psi_k}\{M_\gamma : \gamma\in\Psi_k\}$, where $\{M_\gamma : \gamma\in\Psi_k\}$ is α_1-open neighbourhood whose $\overline{\{W_\epsilon : \epsilon\in\Gamma_k^*\}}$ is α_2-compact. Suppose $H_k(\epsilon,k) = (K,\beta_1,\beta_2) - \Theta((G,\alpha_1,\alpha_2) - \bigcup_{\gamma\in\Psi_k} M_\gamma)$ is a β_1-open set containing k,

$H_k^*(\epsilon,k) = (K,\beta_1,\beta_2) - \Theta((G,\alpha_1,\alpha_2) - \bigcup_{\gamma\in\Psi_k^*}\overline{W_\epsilon : \epsilon\in\Psi_k^*})$ is a β_2-compact set containing k, where $\Theta^{-1}(H_k(\epsilon,k)) \subseteq \bigcup_{\gamma\in\Psi_k} M_\gamma$ is α_1-open neighbourhood whose $\Theta^{-1}\ (H_k^*(\epsilon,k) \subseteq \bigcup_{\gamma\in\Psi_k^*}\overline{W_\epsilon : \epsilon\in\Psi_k^*}$ is α_2-compact. Since (K,β_1,β_2) is pairwise paracompact it has pairwise open locally finite parallel refinement $\underset{\sim}{P} = \{P_B : B\in\Psi_1\}\bigcup\{P_B^* : B\in\Psi_2\}$,where $\{H_B : B\in\Psi_1\}$ is β_1-locally finite Paracompact of H_k and $\{P_B^* : B\in\Psi_2\}$ is β_2-locally finite paracompact of H_k^*, $\Psi = \Psi_1\bigcup\ \Psi_2$. Let $S_1 = \{\Theta^{-1}(P_B)\bigcap W_{\gamma_o}\ , o\ = 1,2,...,n, B\in\Psi_1, \epsilon\in\Gamma_k\}$ is α_{1-} open locally finite parallel refinement of $\{M_\gamma : \gamma\in\Psi_k\}$,and

let $S_2 = \{\Theta^{-1}(P_B^*)\bigcap W_{\epsilon o}\ , o\ = 1,2,...,n,\ B\in\Gamma_2\ ,\ \epsilon\in\Gamma_k^*\}$ is α_2- open locally finite parallel refinement of $\{W_\epsilon : \epsilon\in\Gamma\}$.

Let $\underset{\sim}{S} = \{S_1\bigcup S_2\}$, then $\underset{\sim}{S}$ is pairwise open locally finite parallel refinement of $\underset{\sim}{N}$, so (G,α_1,α_2) is pairwise paracompact space. □

Remark 4.25. Let $\Theta : (G,\alpha_1,\alpha_2) \to (K,\beta_1,\beta_2)$ be a pairwise locally lindelöf proper function, and (K,β_1,β_2) is a pairwise paracompact, then (G,α_1,α_2) is so.

5. Conclusions

In the bitopological spaces that functions generate, this study looked into the relationships between the pairwise locally lindelöf proper function and pairwise locally proper function. The work established the prerequisites for harmonising the covers and the locally discrete spaces in accordance with the notion of pairs locally proper functions thus provided. We looked at the relationship between these two ideas and gave them various coverings to describe them. This study's secondary goal was to emphasize certain intricate characteristics of the paired locally appropriate functions and some peculiarities of the cartesian process of these functions' multiplication in novel contexts.This study looked into the connections between the ideal spaces. Furthermore, key aspects of these concepts as well as a few instructive situations were carefully investigated. We identified their fundamental characteristics in general and made clear the requirements for establishing similar linkages between them. We talked about their main traits and demonstrated how they work together. The report also highlighted the characteristics of these functions and offered numerous instances of them. Investigations into the various potential futures for these functions will begin with these functions. Future studies might look into investigating more variations of these functions.

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Conflict of interest

We certify that we don't have any competing interests.

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